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# A note on "Re-examining the law of iterated expectations for Choquet decision makers"

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**Abstract**

This note completes the main result of [Zimper A., (2010) Re-examining the law of iterated expectations for Choquet decision makers. *Theory and decision*, DOI 10.1007/s11238-010-9221-8], by showing that additional conditions are needed in order the law of iterated expectations to hold true for Choquet decision makers. Due to the comonotonic additivity of Choquet expectations, the equation

$$E[f, \nu(d\omega)] = E[E[f(\omega_{i,j}), \nu(A_{i,j}|A_i)], \nu(A_i)],$$

is valid only when the act $f$ is comonotonic with its dynamic form, that we name "conditional certainty equivalent act".

Keywords: Choquet; Capacities; Updating; Law of iterated expectations

JEL Classification : D81

## 1 Introduction

The Choquet model enlarges the standard approach thanks to a generalized notion of probability, called Choquet capacity. The extension of Choquet expectations to a dynamic set-up is a subject of matter. Whereas several updating rules have been proposed in the literature, no one allows preferences to exhibit a recursive structure as, for instance the multiple priors model (see Epstein and Schneider 2003). Such a feature would be suitable for applications as well as normative reasons.

Zimper (2010) uses the updating rule for Choquet capacities proposed by Sarin and Wakker (1998). He claims that it allows the law of iterated expectations to hold for Choquet decision makers. In this note, we give counter-examples to his theorem. They show that if the criterion is a Choquet expectation with respect to a non-additive capacity, then the law of iterated expectations does not hold in general. Nevertheless, there are particular cases where Zimper's theorem is valid. We identify them.

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## 2 Notations and definitions

Let $\Omega$ be a *state space* endowed with a $\sigma$-algebra denoted by $\mathcal{F}$, so that $(\Omega, \mathcal{F})$ is a measurable space. Elements of $\mathcal{F}$ are called events and for any $A \in \mathcal{F}$, the event $\Omega \backslash A$ is noted $A^c$. We will consider throughout random variables (or acts) $f : \Omega \to \mathbb{R}$ s.t. $f$ is a $\mathcal{F}$-measurable function taking only finite values. The set of such functions is denoted by $\mathcal{A}$. A Choquet capacity is a set function $\nu : \mathcal{F} \to [0, 1]$ such that (i) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and (ii) $\forall A, B \in \mathcal{F}, A \subset B \Rightarrow \nu(A) < \nu(B)$.

In order to define the Choquet expectation of an act $f$ w.r.t. $\nu$, denoted by $E[f, \nu]$ we associate to any r.v. $f \in \mathcal{A}$ the coarsest finite partition over $\Omega$, $\mathcal{P} = \{A_1, ..., A_m\}$, to which $f$ is measurable and ordered. Hence the Choquet expectation of $f \equiv (x_1, A_1; ...; x_m, A_m)$, w.l.o.g. $x_1 < ... < x_m$, can be written as

$$E[f, \nu] = x_1 + \sum_{i=2}^{m} [x_i - x_{i-1}] \cdot \nu(\cup_{j=i}^{m} A_j) \tag{1}$$

$$= \sum_{i=1}^{m} x_i \cdot [\nu(A_i \cup ... \cup A_m) - \nu(A_{i+1} \cup ... \cup A_m)] \tag{2}$$

where $[\nu(A_i \cup ... \cup A_m) - \nu(A_{i+1} \cup ... \cup A_m)]$ is called a *decision weight*. For any $A \in \mathcal{F}$, $D_f[A]$ refers to the dominating event, that is the event that dominates $A$ when the valued act is $f$, such that

$$D_f[A] = \{\omega \in A^c | \forall \omega' \in A, f(\omega) \geq f(\omega')\} \tag{3}$$

Then the Choquet expectation of $f$, with

$$\tilde{\nu}_f(A_i) = \nu(A_i \cup D_f[A_i]) - \nu(D_f[A_i]) \tag{4}$$

can be rewritten as

$$E[f, \nu] = \sum_{i=1}^{m} x_i \cdot \tilde{\nu}_f(A_i) \tag{5}$$

A crucial concept in this model is the comonotonicity.

**Definition 1.** *Two ramdom variables $f$ and $g$ are* comonotonic *if and only if* $\forall \omega, \omega' \in \Omega$, $[f(\omega) - f(\omega')][g(\omega) - g(\omega')] \geq 0$.

On the opposite, $f$ and $g$ are *antimonotonic* if $\geq$ is replaced by $\leq$. An important property of the Choquet expectation is its *comonotonic additivity*, which states that if $f$ is comonotonic with $g$, then

$$E[f, \nu] + E[g, \nu] = E[f + g, \nu] \tag{6}$$

Indeed, in this case, $f$ and $g$ use the same decision weight.

In the Choquet framework, new information is integrated in the decision process by means of an updating rule that specifies the way of calculate the conditional capacity $\nu(.|.)$. For instance, if the decision maker is informed that the "right" state is in event $B$, then the conditional Choquet expectation of a r.v. $f \equiv (x_1, A_1; ...; x_m, A_m)$, w.l.o.g. $x_1 < ... < x_m$, is given by :

$$E[f, \nu(.|B)] = \sum_{i=1}^{m} x_i \cdot [\nu(A_i \cup ... \cup A_m|B) - \nu(A_{i+1} \cup ... \cup A_m|B)] \tag{7}$$

where $\nu(.|B)$ is a conditional capacity. If we use the updating rule proposed by Sarin and Wakker (1998) for the decision weight $\tilde{\nu}_f(.)$, then

$$\tilde{\nu}_f(A|B) = \frac{\nu((A \cap B) \cup D_f[A \cap B]) - \nu(D_f[A \cap B])}{\nu(B \cup D_f[B]) - \nu(D_f[B])} \tag{8}$$

hence

$$E[f, \nu(.|B)] = \sum_{i=1}^{m} x_i \cdot \tilde{\nu}_f(A_i|B) \tag{9}$$

Following Zimper, we consider a two-stage filtration $\mathcal{G} = \{\mathcal{G}_t, t = 0, 1, 2\}$, that is an increasing sequence of events, such that

$$\mathcal{G}_0 = \{\Omega, \emptyset\},\ \mathcal{G}_1 = \{\Omega, \emptyset, A_1, ..., A_m\}$$

and

$$\mathcal{G}_2 = \{\Omega, \emptyset, A_1, ..., A_m, A_{1,1}, ..., A_{1,m_1}, ..., A_{m,1}, ..., A_{m,m_m}\}$$

with $(\cup_{j=1}^{m_i} A_{i,j}) = A_i$.

Finally, recall the definition of the law of iterated expectations for the Choquet case :

**Definition 2.** *Let $f \in \mathcal{A}$ be a $\mathcal{G}_2$-measurable act. The law of iterated expectations holds for $f$ and $\nu$ if and only if*

$$E[f, \nu] = E[E[f, \nu(.|A_i)], \nu(A_i)] \tag{10}$$

To understand this law, consider the act $g$ such that $g(\omega_{i,j}) = E[f, \nu(.|A_i)]$ for all $i = 1, ..., m$ and $j = 1, ..., m_i$. We call such an act the *conditional certainty equivalent act* of $f$. The law of iterated expectations means that the DM does not care about the timing of resolution of uncertainty, such that she is indifferent between the act $f$ and its "dynamic" form. This law is trivially verified by the Bayesian model. Nevertheless, Choquet preferences are generally perceived as unable to perform it. The main result of Zimper (2010) states that Choquet preferences may satisfy the law of iterated expectations if conditional capacities are given by formula (8). In the next section, we show that his result needs additional conditions to hold true.

# 3 Result

We argue that the result of Zimper (2010), stating that the law of iterated expectations is valid for Choquet maximizers when the conditional capacity is derived from formula (8), does not hold in general. It can be seen by means of a simple example.

Let $\Omega = \{\omega_1, ..., \omega_4\}$, with first stage events $A_1 = \{\omega_1, \omega_3\}$ and $A_2 = \{\omega_2, \omega_4\}$. Let $f$ be a random variable such that $\forall i = 1, ..., 4$, $f(\omega_i) = i$ and $\nu(.)$ be a capacity such that :

$$\forall i, j, k = 1, ..., 4,\ i \neq j \neq k,\ \nu(\{\omega_i, \omega_j, \omega_k\}) = \frac{1}{2}$$

$$\forall i, j = 1, ..., 4,\ i \neq j,\ \nu(\{\omega_i, \omega_j\}) = \frac{1}{3}$$

$$\forall i = 1, ..., 4,\ \nu(\{\omega_i\} = \frac{1}{4}$$

The Choquet expectation of $f$ is

$$E[f,\nu] = 1 + \nu(\{\omega_2,\omega_3,\omega_4\}) + \nu(\{\omega_3,\omega_4\}) + \nu(\{\omega_4\}) = \frac{25}{12} \tag{11}$$

and its conditional expectations are

$$E[f,\nu(.|A_1)] \tag{12}$$
$$= \frac{\nu(\{\omega_1\}\cup D_f[\{\omega_1\}]) - \nu(D_f[\{\omega_1\}]}{\nu(A_1\cup D_f[A_1]) - \nu(D_f[A_1])} + 3\cdot\frac{\nu(\{\omega_3\}\cup D_f[\{\omega_3\}]) - \nu(D_f[\{\omega_3\}])}{\nu(A_1\cup D_f[A_1]) - \nu(D_f[A_1])} \tag{13}$$
$$= \frac{1-\nu(\{\omega_3\}\cup A_2)}{\nu(A_1\cup\{\omega_4\}) - \nu(\{\omega_4\})} + 3\cdot\frac{\nu(\{\omega_3,\omega_4\}) - \nu(\{\omega_4\})}{\nu(A_1\cup\{\omega_4\}) - \nu(\{\omega_4\})} = 3 \tag{14}$$

and

$$E[f,\nu(.|A_2)] = 2\cdot\frac{\nu(\{\omega_3\}\cup A_2) - \nu(\{\omega_3,\omega_4\})}{\nu(A_2)} + 4\cdot\frac{\nu(\{\omega_4\})}{\nu(A_2)} = 4 \tag{15}$$

Therefore, the unconditional expectation of the conditional certainty equivalent act is

$$E[E[f,\nu(.|A_i)],\nu(A_i)] = 3 + \nu(A_2) = \frac{10}{3} \tag{16}$$

and then, from eq. (11) and (16),

$$E[f,\nu] \neq E[E[f,\nu(.|A_i)],\nu(A_i)] \tag{17}$$

in contradiction with the law of iterated expectations.

Furthermore, it is straightforward that:

**Remark 1.** *The formula (8) is not an update rule for Choquet capacities.*

Stated otherwise, the conditional decision weight obtained by applying the Sarin and Wakker update to $\tilde{\nu}_f(.)$ may be superior to 1, hence the conditional set function $\nu(.|A_i)$ is not normalized thus it is not a Choquet capacity. To see it, consider eq. (14) in the previous example and observe that

$$\tilde{\nu}_f(\{\omega_1\}|A_1) = \frac{1-\nu(\{\omega_3\}\cup A_2)}{\nu(A_1\cup\{\omega_4\}) - \nu(\{\omega_4\})} > 1$$

The previous example shows that the law of iterated expectations does not necessarily holds if the capacity is not additive. Nevertheless, it does for Choquet expectations in some cases. Specifically, it turns out to be when $f$ and its conditional certainty equivalent act are comonotonic. Indeed, in this case, the same decision weight will be used to value these two random variables. Consider again the previous example and let $B_1 = \{\omega_1,\omega_2\}$ and $B_2 = \{\omega_3,\omega_4\}$ be first stage events. Then, conditional expectations of $f$ are

$$E[f,\nu(.|B_1)] = \frac{1-\nu(\{\omega_2,\omega_3,\omega_4\})}{1-\nu(B_2)} + 2\cdot\frac{\nu(\{\omega_2,\omega_3,\omega_4\}) - \nu(B_2)}{1-\nu(B_2)} = \frac{5}{4} \tag{18}$$

and

$$E[f,\nu(.|B_2)] = 3\cdot\frac{\nu(B_2) - \nu(\{\omega_4\})}{\nu(B_2)} + 4\cdot\frac{\nu(\{\omega_4\})}{\nu(B_2)} = \frac{15}{4} \tag{19}$$

hence the conditional certainty equivalent act of $f$ is $g$ such that:

$$\begin{cases} g(\omega) = 5/4 \quad when \quad \omega \in B_1; \\ g(\omega) = 15/4 \quad when \quad \omega \in B_2. \end{cases}$$

The Choquet expectation of $g$ is $E[g, \nu] = 25/12$ and it is equal to the one of $f$. Further, the Sarin and Wakker update rule reduces to the Dempster-Shafer (pessimistic) update rule

$$\nu(A|B_1) = \frac{\nu((A \cap B_1) \cup B_2) - \nu(B_2)}{1 - \nu(B_2)} \tag{20}$$

since $D_f[A \cap B_1] = D_f[B_1] = B_2$ when $A = \{\omega_2, \omega_3, \omega_4\}$, or to the Bayes (optimistic) update rule

$$\nu(A|B_2) = \frac{\nu(A \cap B_2)}{\nu(B_2)} \tag{21}$$

since $D_f[A \cap B_2] = D_f[B_2] = \emptyset$ when $A = \{\omega_4\}$. It implies that the conditional set function $\nu(.|A_i)$ is a Choquet capacity. This result is linked to the one of Chateauneuf et al. (2001). They showed that if $f$ and its conditional certainty equivalent act $g$ are comonotonic, then Choquet expectations have a recursive structure if the DM uses the optimistic update rule conditionally to the "good" event and the pessimistic update rule conditionally to the "bad" event. It can be seen as an extension of the $f$-Bayesian approach of Gilboa and Schmeidler (1993).

More generally, let $f \equiv (x_{1,1}, A_{1,1}; ...; x_{m,m}, A_{m,m})$ and $x_{1,1} < ... < x_{m,m_m}$. In this case, $f$ is comonotonic with its conditional certainty equivalent act $g$ since $E[f, \nu(.|A_1)] < ... < E[f, \nu(.|A_m)]$ by monotonicity of conditional Choquet expectations. The Choquet expectation of $f$ is

$$E[f, \nu] = \sum_{i=1}^{m} \sum_{j=1}^{m_i} x_{i,j} \cdot \tilde{\nu}_f(A_{i,j}) \tag{22}$$

and the Choquet expectation of $g$, that is a constant act on each $A_i$ ($g(\omega_{i,1}) = ... = g(\omega_{i,m_i})$), is

$$\begin{aligned} E[g, \nu] &= \sum_{i=1}^{m} g(\omega_i) \cdot \tilde{\nu}_g(A_i) \\ &= \sum_{i=1}^{m} E[f, \nu(.|A_i)] \cdot \tilde{\nu}_g(A_i) \\ &= \sum_{i=1}^{m} [\sum_{j=1}^{m_i} \frac{\tilde{\nu}_f(A_{i,j})}{\tilde{\nu}_f(A_i)} \cdot x_{i,j}] \cdot \tilde{\nu}_g(A_i) \end{aligned}$$

By comonotonic additivity of the Choquet integral, $\tilde{\nu}_f(A_i) = \tilde{\nu}_g(A_i)$ for all $i = 1, ..., m$, hence $E[f, \nu] = E[g, \nu]$. We state it generally:

**Theorem 1.** *If any $\mathcal{G}_2$-measurable function $f : \Omega \to \mathbb{R}$ is comonotonic with its conditional certainty equivalent act $g(.)$ such that*

$$g(\omega_{i,j}) = E[f, \nu(.|A_i)]$$

*for all $i = 1, ..., m$ and $j = 1, ..., m_i$, then, under the Sarin and Wakker update rule, the law of iterated expectations holds.*

*Proof.* It is sufficient to observe that the proof of Zimper (2010) holds true for these cases, as illustrated before. □

This theorem explains why the example of Zimper (in section 4) works properly. It is implicit in his paper that the comonotonic condition between the valued act and the conditional certainty equivalent act holds. Such a condition is called "nest-monotonicity" by Koida (2010).

# 4 Conclusion

From an axiomatic point of view, the claims of this note can be explained by the inability of Choquet expectations to simultaneously satisfy *consequentialism* and *dynamic consistency* (see Lapied and Toquebeuf 2010). Both axioms seem to be necessary to the law of iterated expectations to universally hold. But then the result of Yoo (1991) holds, too, and the capacity has to be additive.


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